MATHEMATICS
4º ESO

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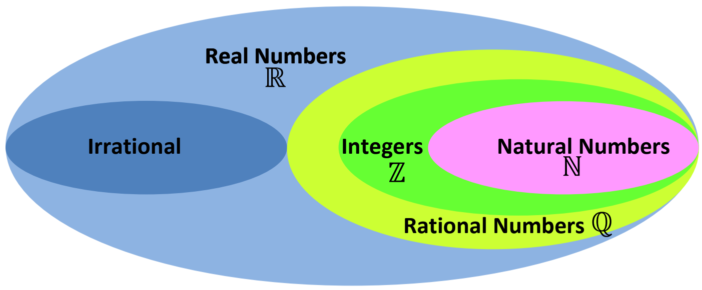
LESSON 1: REAL NUMBERS

Keywords

REAL diagonal RATIONAL irrational radicand golden section divine proportion INTERVAL SINGLE phi root Thales' theorem prove-proof endpoint radical FRACTION non-ending Set-builder parenthesis indices infinity symbol SCIENTIFIC NOTATION rationalize

1. NUMBERS

So far, you have studied different sets of numbers: natural numbers, integers, fractions or rational numbers and real numbers.



Remember: A ***set*** is a collection of objects, these objects are called ***elements***.

- *Natural (or counting) Numbers* (ℕ):0, 1, 2, 3, 4 …
- *Integers* (ℤ): …, -3, -2, -1, 0, 1, 2, 3, …
- *Rational numbers* (ℚ): Natural numbers, integers and fractions are **rational** numbers. A **fraction** is formed by two numbers like:

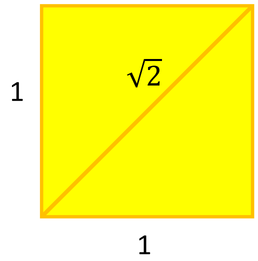
$$\frac{5}{3}$$ 5 ← Numerator, 3 ← Denominator

- *Irrational numbers*: They are numbers which can't be expressed as fractions. $\sqrt{2}, \sqrt{3}, \pi, \phi \ldots$
- *Real numbers* (ℝ): It's the set formed by rational and irrational numbers.

1.1. Irrational numbers

You know a lot about rational numbers, so we are going to study irrational numbers more deeply in this lesson.

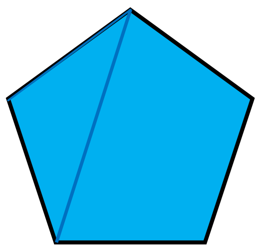
- $\sqrt{2}$ is an irrational number, so it can't be expressed as a fraction. Let's prove it!



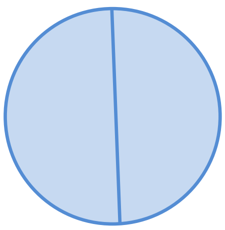
<u>*Proof:*</u> Let's suppose that $\sqrt{2}=\frac{p}{q}$ (proof by contradiction), then $2=\frac{p^2}{q^2} \Rightarrow p^2=2q^2 \cdot p^2$ is a perfect square and it's also an even number, so <u>it contains the factor 2 an even number of times</u>, and then q^2 must be even too and it contains the factor 2 an even number of times, but taking into account that $p^2=2q^2$ we get that p^2 <u>has the factor 2 an odd number of times</u>. Finally there is a contradiction in the underlined sentences because $\sqrt{2}$ can't be written as a fraction.

- The ***golden*** number $\phi = \frac{1+\sqrt{5}}{2}$ is an irrational number. It's also known as ***golden section*** and ***divine proportion***.

  The ratio between the diagonal and the side of a regular pentagon is ϕ.



- π is the relation between the length of a circumference and its diameter.

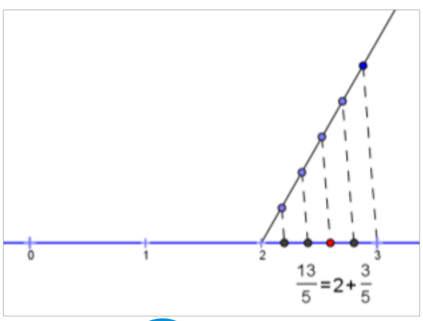


$$L = 2\pi r = \pi d \quad \Rightarrow \quad \pi = \frac{L}{d}$$
$$d = diameter, \quad r = radius$$
$$L = length$$

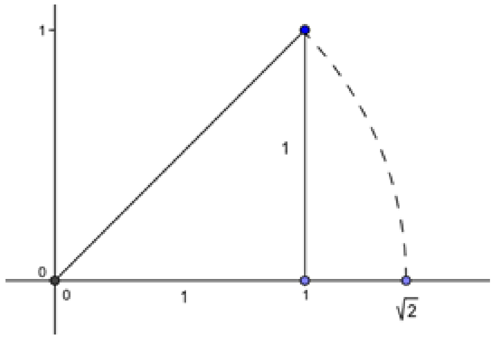
1.2. Number line

We can represent numbers on the real number line.

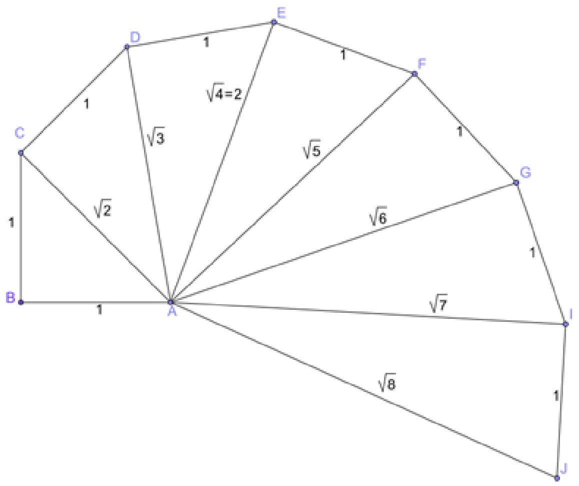
- Rational numbers: you can represent them using Thales' Theorem.



- Irrational numbers: you can represent some of them using Pythagoras' Theorem. Look at this example:



Exercise: Describe this picture where every triangle is a right triangle.



2. INTERVALS

An interval is a set formed by the real numbers between, and sometimes including, two numbers. They can also be non-ending intervals as we are going to see.

(	means "not included" or "open"
[	means "included" or "closed"

- Open interval **(1, 3)**, formed by all the real numbers between 1 and 3, but the ***endpoints*** are **not included**.

$$(1,3)=\{x \in \mathbb{R} : 1 < x\ 3\}$$

*Note: The notation between {} is called **set-builder** notation.*

- Closed interval **[1, 3]**, formed by all the numbers between 1 and 3 where the ***endpoints*** are **included**.

$$[1,3]=\{x \in \mathbb{R} : 1 \leq x \leq 3\}$$

- Half-open interval (left-closed, right-open) **[1, 3)** which contains the real numbers between 1 and 3. 1 is included but 3 is not.

$$[1,3)=\{x \in \mathbb{R} : 1 \leq x < 3\}$$

LESSON 1

- Half-open interval (left-open, right-closed) **(1, 3]** which contains the real numbers between 1 and 3. 1 is not included but 3 is included.

$$(1,3]=\{x \in \mathbb{R} : 1 < x \leq 3\}$$

Some intervals don't end, they are called ***non-ending*** intervals. One (or both) endpoints are $+\infty$ and $-\infty$. We always use the parenthesis "**(**" with these symbols. There are different possibilities:

- **(1, +∞)=** $\{x \in \mathbb{R} : x > 1\}$ Observe that 1 is not included.
- **(−∞, 3)=** $\{x \in \mathbb{R} : x < 3\}$ Observe that 3 is not included.
- **[1, +∞)=** $\{x \in \mathbb{R} : x \geq 1\}$ Observe that 1 is included.
- **(−∞, 3]=** $\{x \in \mathbb{R} : x \leq 3\}$ Observe that 3 is included.
- **(−∞, +∞)=** $\mathbb{R}$

Exercise: Express these intervals using set-builder notation.

a) [-4, 3]={ }

b) (2, 6]= { }

c) (−∞, 4)= { }

d) [0, +∞)= { }

3. RADICALS

Remember:

$$\sqrt[n]{a} = b \Rightarrow b^n = a$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

Properties of radicals

1. The n-th root of a negative number doesn't exist if n is an even number.

   Example: $\sqrt[4]{-81}$ doesn't exit.

2. The n-th root, where n is an even number, of a positive number has two different solutions, one is positive and the other one is negative.

   Example: $\sqrt{16} = \pm 4$.

3. You can simplify radicals expressing them as powers.

   Example: $\sqrt[4]{25} = \sqrt[4]{5^2} = 5^{\frac{2}{4}} = 5^{\frac{1}{2}}$.

4. You can convert radicals with different indices into radicals with the same index. First, express the radicals as powers. Secondly, convert the fractions into fractions with the same denominator.

   Example: $\sqrt{7}$ and $\sqrt[3]{10} \Longrightarrow 7^{\frac{1}{2}}$ and $10^{\frac{1}{3}} \Longrightarrow 7^{\frac{3}{6}}$ and $10^{\frac{2}{6}} \Longrightarrow \sqrt[6]{7^3}$ and $\sqrt[6]{10^2}$

5. You can multiply radicals with the same index.

   Example: $\sqrt[3]{5} \cdot \sqrt[3]{7} = \sqrt[3]{5 \cdot 7} = \sqrt[3]{35}$

6. You can divide radicals with the same index.

   Example: $\frac{\sqrt[4]{45}}{\sqrt[4]{9}} = \sqrt[4]{\frac{45}{9}} = \sqrt[4]{5}$

7. You can multiply and divide radicals with different indices. First, you need to convert them into radicals with the same index.

   Example: $\sqrt[3]{5} \cdot \sqrt[4]{2} = \sqrt[12]{5^4} \cdot \sqrt[12]{2^3} = \sqrt[12]{5^4 \cdot 2^3}$

8. To calculate the power of a root you raise the radicand to the power.

   Example: $\left(\sqrt[3]{2}\right)^5 = \sqrt[3]{2^5}$

9. You can simplify roots factoring the radicand and "taking out" the factors whose exponents are equal or greater than the index. Divide the exponent by the index and the quotient is the exponent of the factor you can "take out", the remainder is the exponent of the factor inside the radical. It will be easier to understand with this example.

   Example: $\sqrt[3]{2^5 \cdot 3^8 \cdot 5^9} = 2 \cdot 3^2 \cdot 5^3 \cdot \sqrt[3]{2^2 \cdot 3^2}$

10. To calculate the root of a root you multiply the indices.

    Example: $\sqrt[5]{\sqrt[3]{2}} = \sqrt[15]{2}$

11. To add or subtract radicals they must have the same index and the same radicand.

    Example: $5\sqrt[3]{2} + 3\sqrt[3]{2} - \sqrt[3]{2} = 7\sqrt[3]{2}$

12. Sometimes in algebra we want to find an equivalent expression for a radical expression that doesn't have any radicals in the denominator. This process is called ***rationalizing*** the denominator.

    The process consists of multiplying the numerator and the denominator by the same expression. There are three different cases:

    - Single square root

      Example:

      $\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{5^2}} = \frac{2\sqrt{5}}{5}$ You multiply numerator and denominator by $\sqrt{5}$.

- Single higher root

Example:

$\frac{3}{\sqrt[3]{2}}=\frac{3}{\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}=\frac{3\sqrt[3]{2^2}}{\sqrt[3]{2^3}}=\frac{3\sqrt[3]{2^2}}{2}$ You multiply numerator and denominator by $\sqrt[3]{2^2}$.

- Sums and differences of square roots. Multiply top and bottom by the difference, if the original is a sum and by the sum, if the original is a difference. This way you will get the *difference of squares* and get rid of the square roots.

Examples:

a) $\frac{5}{1+\sqrt{2}}=\frac{5}{1+\sqrt{2}}\cdot\frac{1-\sqrt{2}}{1-\sqrt{2}}=\frac{5\cdot(1-\sqrt{2})}{(1+\sqrt{2})\cdot(1-\sqrt{2})}=\frac{5-5\sqrt{2}}{1^2-(\sqrt{2})^2}=\frac{5-5\sqrt{2}}{1-2}=\frac{5-5\sqrt{2}}{-1}=$

$=5\sqrt{2}-5$

b) $\frac{\sqrt{3}}{2-\sqrt{7}}=\frac{\sqrt{3}}{2-\sqrt{7}}\cdot\frac{2+\sqrt{7}}{2+\sqrt{7}}=\frac{\sqrt{3}\cdot(2+\sqrt{7})}{(2-\sqrt{7})\cdot(2+\sqrt{7})}=\frac{2\sqrt{3}+\sqrt{21}}{2^2-(\sqrt{7})^2}=\frac{2\sqrt{3}+\sqrt{21}}{4-7}=\frac{2\sqrt{3}+\sqrt{21}}{-3}$

4. SCIENTIFIC NOTATION

Scientific notation (also known as standard form) is used for numbers that are either very large or very small, like:

- The mass of the Earth is about 6 000 000 000 000 000 000 000 000 kg
- X-rays have a wavelength of about 0,000 000 095 cm

Scientific notation is a more convenient way of expressing such numbers for working with calculators and computers.

A number in scientific notation is expressed as

$$N = a{,}bcdef\ldots \cdot 10^n$$

So,

- The mass of the Earth is about $6 \cdot 10^{24}$ kg
- X-rays have a wavelength of about $9{,}5 \cdot 10^{-8}$ cm

Let's see how to calculate with numbers expressed in scientific notation:

✓ Addition and subtraction: the numbers must be expressed with the same exponent.

Example:

$2{,}13\cdot10^2+5{,}36\cdot10^4=2{,}13\cdot10^2+536\cdot10^2=538{,}13\cdot10^2=5{,}3813\cdot10^4$

✓ Multiplication and division: you multiply or divide the decimal numbers and add or subtract the exponents.

Examples:

a) $(1{,}45\cdot10^3)\cdot(2{,}06\cdot10^5)=2{,}987\cdot10^8$

b) $(3{,}21\cdot10^7):(5{,}81\cdot10^4)=0{,}5525\cdot10^3=5{,}525\cdot10^2$

PRONUNCIATION

- Diagonal |daɪ'ægənəl|
- Divine proportion |dɪ'vaɪn prə'pɔːʃən|
- Endpoint |'end'pɔɪnt|
- Fraction |'frækʃən|
- Golden section |'gəʊldən 'sekʃən|
- Indices |'ɪndɪsiːz|
- Infinity symbol |ɪn'fɪnɪtɪ 'sɪmbəl|
- Interval |'ɪntəvəl|
- Irrational |ɪ'ræʃənəl|
- Non-ending |nɒn 'endɪŋ|
- Parenthesis |pə'renθəsɪs|
- Phi |'faɪ|
- Prove-proof |pruːv pruːf|
- Radical |'rædɪkəl|
- Radicand |'rædɪˌkænd|
- Rational |'ræʃnəl|
- Rationalize |'ræʃnəlaɪz|
- Real |riːl|
- Root |ruːt|
- Scientific notation |ˌsaɪən'tɪfɪk nəʊ'teɪʃən|
- Set-builder |set 'bɪldə|
- Single |'sɪŋgəl|
- Thales' theorem |'θeɪliːz 'θɪərəm|

WORKSHEET

1. Represent $\sqrt{17}$ and $\sqrt{34}$ on the number line.

2. Express these intervals using set-builder notation. Represent them on the number line:
   a) $[2, 5)=$
   b) $(-2, +\infty)=$
   c) $(-3, 8)=$
   d) $(-\infty, -6)=$

3. Simplify:
   a) $\sqrt[3]{2}\cdot\sqrt[3]{4}=$
   b) b) $\sqrt[4]{3}\cdot\sqrt[3]{5}=$
   c) $\dfrac{\sqrt[5]{a^4\cdot b^3\cdot c^6}}{\sqrt[4]{a\cdot b^2\cdot c^3}}=$
   d) $\left(\sqrt[3]{x^2}\right)^9=$
   e) $\sqrt[3]{\sqrt{\sqrt[4]{2^{24}}}}=$
   f) $2\sqrt[3]{7}-\sqrt[3]{56}+3\sqrt[3]{189}=$
   g) $\sqrt[3]{864a^4b^{10}}=$

4. Rationalize these expressions:
   a) $\dfrac{2}{\sqrt{11}}=$
   b) $\sqrt[3]{\dfrac{2}{3}}=$
   c) $\dfrac{3}{3-\sqrt{5}}=$
   d) $\dfrac{1}{\sqrt{2}+\sqrt{3}}=$

5. Calculate and express the result in scientific notation:
   a) $2{,}32\cdot10^5-1{,}47\cdot10^7=$
   b) $(5{,}16\cdot10^4)\cdot(4{,}29\cdot10^5)=$
   c) $(3{,}67\cdot10^6):(9{,}15\cdot10^2)=$
   d) $7{,}33\cdot10^9-8{,}09\cdot10^6=$

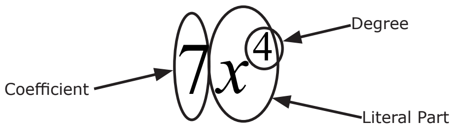
LESSON 2: POLYNOMIALS AND ALGEBRAIC FRACTIONS

Keywords

MONOMIAL coefficient DEGREE **leading term** Polynomial quotient REMAINDER *binomial* **EVALUATE** Ruffini's Rule **FACTORING** TRINOMIAL the simplest form THEOREM **algebraic fraction** CANDIDATE *root* divisor

1. MONOMIALS

As you know, a **monomial** is an algebraic expression consisting of only one term, which has a known value (coefficient) multiplied by one or some unknown values represented by letters with exponents that must be constant and positive whole numbers (literal part). For example:



If the literal part of a monomial has only one letter, then the degree is the exponent of the letter. If the literal part of a monomial has more than one letter, then the degree is the addition of the exponents of the letters.

Examples:

The degree of $-5x^3$ is 3.

The degree of $-7x^2y^3$ is $2+3=5$.

Addition and subtraction of monomials

You can add monomials *only if they have the same literal part* (they are also called **like terms**). In this case, you *add the coefficients and leave the same literal part*.

Examples:

a) $3x+2x=5x$

b) $3x+2x^2$ (You cannot add the terms $3x$ and $2x^2$ because these are not like terms).

c) $5x^2+7-3x^2-4=2x^2+3$ (You cannot add the terms $2x^2$ and 3).

Multiplication of monomials

If you want to multiply two or more monomials, you just have to multiply the coefficients, and add the exponents of the equal letters:

Examples:

a) $2x^7 \cdot 3x^3 = 6x^{10}$

b) $(2xy^3)\cdot(-3xy^2) = -6x^2y^5$

Division of monomials

If you want to divide a monomial by a monomial of the same or lower degree, you just have to divide the coefficients, and subtract the exponents of the equal letters.

Examples:

a) $(10x^5)\div(-2x^2) = -5x^3$

b) $(12a^2b)\div(3a) = 4ab$

2. POLYNOMIALS

A **polynomial** is the addition or subtraction of two or more monomials (which are called terms).

-If there are two monomials, it is called a binomial, for example $x^2 + x$.

-If there are three monomials, it is called a trinomial, for example $3x^2 - 5x + 11$

The degree of the polynomial is the highest degree of the terms that it contains.

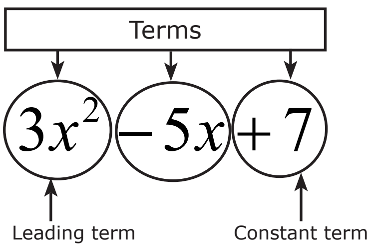
Examples:

a) $3x^2 - 5x + 11$ is a second-degree trinomial.

b) $x^4 - 5x$ is a fourth-degree binomial.

Example:

The following polynomial is a second-degree polynomial, and contains three terms: 3x² is the leading term (the term with the highest exponent), and 7 is the constant term.



You usually write polynomials with the terms in "decreasing" order of exponents. We say that a polynomial is **complete** if it has terms of every exponent from the degree of the polynomial until zero.

Examples:

a) The polynomial $3x^2 - 5x + 7$ is complete.

b) The polynomial $3x^2 + 7$ is not complete.

Evaluating polynomials

"Evaluating" a polynomial $P(x)$ is calculating its numberical value at a given value of the variable: $x = a$. You must substitute the variable x for the value a, and calculate the value of the polynomial $P(a)$.

Example: Evaluate $P(x) = 2x^3 - x^2 - 4x + 5$ at $x = -2$

Substitute x for -2 and calculate. But be careful with brackets and negative signs!

$$P(-2) = 2\cdot(-2)^3 - (-2)^2 - 4\cdot(-2) + 5 = 2\cdot(-8) - 4 + 8 + 5 = -16 - 4 + 8 + 5 = -20 + 13 = -7$$

Addition of polynomials

When adding polynomials you must add each like term of the polynomial, that is, monomials that have the same literal part. (You must use what you know about the addition of monomials).

Example: Simplify $\left(3x^3 - 2x^2 + 5x - 3\right) + \left(2x^3 + 4x^2 - 2x - 4\right)$

$$\left(3x^3 - 2x^2 + 5x - 3\right) + \left(2x^3 + 4x^2 - 2x - 4\right) = 5x^3 + 2x^2 + 3x - 7$$

Subtraction of polynomials

When subtracting polynomials you must realise that a subtraction is the addition of the first term and the opposite of the second: A – B = A + (– B)

Example: Simplify: $\left(3x^3 - 2x^2 + 5x - 3\right) - \left(2x^3 + 4x^2 - 2x - 4\right)$

$$\left(3x^3 - 2x^2 + 5x - 3\right) - \left(2x^3 + 4x^2 - 2x - 4\right) =$$

$$3x^3 - 2x^2 + 5x - 3 - 2x^3 - 4x^2 + 2x + 4 = x^3 - 6x^2 + 7x + 1$$

Multiplication of polynomials

-**A Monomial times a multi-term polynomial.** To do this, we have to expand the brackets:

Example: Simplify $-2x(5x^2 - x + 10)$

$$-2x(5x^2 - x + 10) = -2x\cdot\left(5x^2\right) - 2x\cdot\left(-x\right) - 2x\cdot\left(10\right) = -10x^3 + 2x^2 - 20x$$

-**A Multi-term polynomial times a multi-term polynomial**. Look at this example:

Example: Simplify $\left(x + 5\right)\cdot\left(x + 2\right)$

$$\left(x + 5\right)\cdot\left(x + 2\right) = x\left(x\right) + x\left(2\right) + 5\left(x\right) + 5\left(2\right) = x^2 + 2x + 5x + 10 = x^2 + 7x + 10$$